An exploratory study on students' reasoning about symmetry

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The concept of symmetry is vital for the study of science and developing spatial and geometric relationships. Yet few studies have been conducted to examine how students learn this concept. This study investigates 757 Year 4 to 10 students' ability to reason about symmetrical relationships. By analysing the semiotic processes through the lens of the theory of semiotic mediation and Sfards' mathematical discourse framework, the results show the synergic relations between visualisation of artefacts, and the use of signs and keywords in the construction of knowledge.

The word symmetry, or 'symmetros' from Greek, is an assimilation of two ideas, 'syn' meaning together and 'metron' meaning to measure. It denotes being in agreement in dimensions, due proportion and arrangement. This idea of being in harmony of proportions is an essential component that governs all laws of nature, biology and art (Weyl, 1952). Symmetry is commonly associated with an understanding of properties and hierarchies of space in geometry. As a concept, it connects between various branches of mathematics and when used in problem solving situations, it provides a more beautiful and elegant solution (Leikin, Berman, & Zaslavsky, 2000). The concept of invariance, where the properties of a configuration remain unchanged during transformations, is an important component of symmetry and is a key element in the study of physics and chemistry (Anderson, 1972). In this technological world we live in, knowledge of symmetry is an integral part of spatial and geometric reasoning and contributes to the learning of STEM related disciplines.

In Australia, the concept of symmetry is taught from Year 3 to 7 and is based on the premise that teachers are able to make the necessary connection between and across content strands and teach for mathematical reasoning. For example, 'identify symmetry' is addressed in Year 3, 5 and 7 with the assumption that teachers recognise the different emphasis required for each year level. In Year 3, it is about symmetry in the environment. In Year 5, it relates to how shapes are being transformed while in Year 7, it is about identifying symmetries after transformations on an axis and on the Cartesian plane using coordinates. Evidence suggests there is a developmental progression in one's ability to visualise and use symmetrical relationships (Schuler, 2001) and that many concepts of symmetry are not firmly established before the age of 12 (Genkins, 1975). Discrimination of reflectional symmetry appears to move from vertical in 4-year-olds, to horizontal in 5-year-olds and vertical, horizontal and oblique symmetry in 6-year-olds (Bornstein & Stiles-Davis, 1984). However, much of the research in this field is dated. Available evidence shows that through careful semiotic mediation of instructional tools, language and gestures between the teacher and learners, children as young as Year 1 and 2 are capable of learning about reflectional symmetry (Ng & Sinclair, 2015). No study was found on how learners comprehend rotational symmetry.

Research on learning progressions has shown that the provision of evidence-based frameworks, validated tools and quality instructional materials has the potential to transform the teaching and learning of mathematics (Siemon et al., 2018). In this paper, we design, survey and document Australian middle years students' knowledge and ability to reason about reflective and rotational symmetry in order to better comprehend their conceptual knowledge of symmetry and ability to use this knowledge in problem situations. This

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information can then contribute to building sustainable integrated learning and teaching resources to support the development of mathematical reasoning.

Theoretical Framework

Reasoning about symmetry is not merely about noticing if a shape is symmetrical. Rather, it is having a deep understanding and an appreciation of the concept and its relationships with a connected web of other concepts, operations and relations. To reason is to logically reflect, explain and justify a position (Kilpatrick, Swafford, & Findell, 2001). It requires secure conceptual and procedural knowledge and a capacity to communicate reasoning and solution strategies in multiple ways (i.e., through diagrams, symbols, orally and in writing) (Siemon et al., 2018). These elements are developed through five levels of reasoning: visualising, describing, analysing, and inferring relationships and formal deductive proof (Seah & Horne, 2019). The levels are interconnected and develop progressively with various degrees of emphasis and importance depending on the task demand.

From a cognitive perspective, developing a conceptual understanding is dependent on the connectedness between multiple representations, visualisation, and mathematical discourse. Since mathematical concepts are abstract entities, the use of symbols, points, lines, angles, shapes and concrete materials are needed to represent that knowledge. These representations, developed over time, may act as artefacts or signs in the construction of mathematical knowledge. An artefact is a tool or an instrument that relates to a specific task to be used for a particular purpose, such as a printed shape or the net of a cube. A sign is a product of a conjoint effort between it and the mind to communicate an intent, such as a ‘ll’ sign indicating equal side lengths of a shape. Learning mathematics is essentially a social act, directed by a semiotic process (Bartolini Bussi & Mariotti, 2008). In semiotic activities, various artefacts and signs are produced. The artefacts used in geometry are initially understood purely from visual recognition and discussed using informal language. As individuals’ knowledge deepens, so does their ability to look beyond the physical images to infer and deduce the relationships the images present using formal terms and signs to justify a position. As such, the signs are intentional and highly subjective. They link to the learner’s specific experience with the artefact and the task to be carried out. Making sense of mathematical concepts then, requires one to visualise and communicate the multiple representations used to express the idea.

As a cognitive process, visualisation plays a key role in helping individuals to interpret what they see within the network of their personal beliefs, experiences and understandings. To understand reflective symmetry, one must be able to see an ‘imaginary’ mirror line that divides a shape/object so that half of the image is the same as the other half but reflected. To comprehend rotational symmetry, one must locate the centre of the turn, recognise that a visualisation object remains invariant after a partial turn, and that it is different from reflective symmetry. Yet visualisation alone is insufficient as many people only see what they know and form what Vinner (1991) termed as a personal concept image - the collective mental pictures, the corresponding properties and processes that are associated with the concept. To align a personal concept image to its corresponding formal concept definition – a form of words used specific to that concept, constant interactions between visualisation of multiple representations and communication of the relationship is needed. Indeed, thinking and talking are inseparable processes. Learning happens where there is a change in discourse in terms of *keywords*, *visual mediators*, *narratives* and *routines* (Sfard, 2008). In a mathematical discourse, *keywords* and *visual mediators* are used in distinctly mathematical ways. Both trigger images and emotions that are stored in memory, laden with personal beliefs and experiences and shape one's thinking. *Keywords* are used to form descriptions and definition of a concept. *Visual mediators* in the form of artefacts and signs serve as a focal point in the discussion. They are the means by which participants of a discourse identify the object of their talk and coordinate their communication. The *narratives* and *routines* carried out in a mathematical classroom further create sociomathematical norms through which mathematics is learned.

In this study, we seek to determine students' conceptual knowledge of symmetry by looking at the keywords, and signs they used to justify a symmetric situation. Bartolini Bussi and Mariotti's (2008) theory of semiotic mediation and Sfard's (2008) mathematical discourse framework offer a way to understand the relationships among artefacts, the actions they allow one to accomplish, and how individuals use them to construct mathematical concepts.

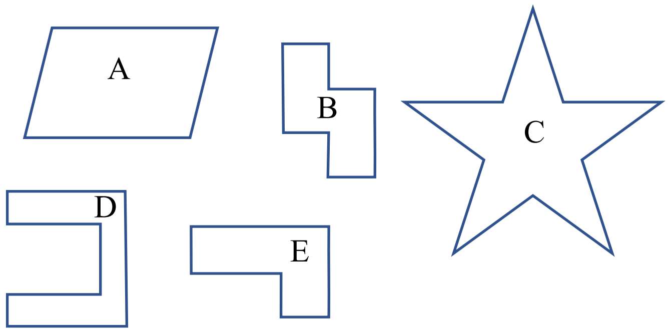
Method

Symmetry

Look at the shapes below

a [GSYM1]

On each of these shapes draw all lines of symmetry.



b [GSYM2]

For each of the shapes in part **a**, decide whether there is any reflectional or rotational symmetry and write the letters in the appropriate space in the table below.

	Has rotational symmetry	**Does not have rotational symmetry**
Has reflectional symmetry		
Does not have reflectional symmetry		

c [GSYM3]

How do you know if a shape has rotational symmetry?

Figure 1. The symmetry task (GSYM).

This paper reports a set of data taken from the Reframing Mathematical Futures II project (Siemon et al., 2018). We apply an iterative cycle of designing, testing, and re-designing an assessment task (Figure 1) and scoring rubric (Figure 2) in order to validate the items. The task, coded as GSYM to mean Geometry Symmetry, assesses students' knowledge of reflectional and rotational symmetry. GSYM1 presents students with 5 shapes and asks them to draw the lines of symmetry. Item A and B have rotational symmetry but no reflectional symmetry. Item C has both rotational and reflectional symmetry. Item D has reflectional symmetry but no rotational symmetry, and item E has no symmetry. GSYM2 requires students to analyse these 5 given artefacts and classify the shapes according to the criteria set out in the table. Students are then asked to explain when a shape has rotational symmetry in GSYM3. The larger data set was analysed based on a Rasch Model for the purpose of verifying the scoring rubric and constructing a learning progression. In this article, we analyse the data based on Bartolini et al.'s theory of semiotic mediation and Sfard's mathematical discourse framework to allow the depth of student knowledge in relation to reasoning about symmetry to be studied in depth.

SCORE	DESCRIPTION for GSYM1
0	No response or irrelevant response
1	No shapes (2D) having all lines drawn correctly
2	All symmetry line(s) drawn correctly on one shape (others may be incorrect)
3	Correct lines drawn on C and D but incorrect lines drawn on at least one other shape
4	D: one line correctly drawn horizontally through centre C: five lines drawn from each point to opposite reflex angled corner No lines drawn on A, B or E
SCORE	**DESCRIPTION for GSYM2**
0	No response or irrelevant response
1	Only one shape (letter) correctly placed
2	At least 2 correctly placed
3	At least 4 correctly placed
4	C has both rotational and symmetrical symmetry A, B have rotational symmetry but no reflectional symmetry D has reflectional symmetry but no rotational symmetry E has no symmetry
SCORE	**DESCRIPTION for GSYM3**
0	No response or irrelevant response
1	An attempt at an explanation but lacking clarity
2	Some explanation about turning shape part way around circle and it looks the same – perhaps around a pin

Figure 2. The symmetry task (GSYM) marking rubric.

The participants were middle-years students from across Australian States and Territories. Two groups of cohorts were involved. The first set of data – the trial data, was taken from 296 Year 4 - 10 students from three primary and four high schools across social strata and four States to allow for a wider spread of data being collected. The teachers administered the assessment tasks and returned the student work to the researchers. The trial results were marked by two markers and validated by a team of researchers to ascertain the usefulness of the scoring rubric and the accuracy of the data entry. The second set of data – the project data, was taken from 461 Year 7 - 10 students from nine high schools situated in lower socioeconomic regions with diverse populations across five States and one Territory. The project school teachers marked the items and returned the raw score instead of individual forms to the researchers. The project school teachers received two 3-day face-to-face professional learning sessions on spatial and geometric reasoning prior to the implementation of the assessment tasks. They also had access to a bank of teaching resources and four on-site visits to support their teaching effort.

Findings

Table 1 shows the overall percentage breakdown of student responses for GSYM. Although the project schools' performance was slightly better than the trial schools, both cohorts' knowledge of symmetry was poor. There were large numbers of no responses or irrelevant responses. Less than 3% correctly drew lines of symmetry for item C and D and only around 1% were able to put all the items in the correct place for GSYM2.

Table 1
Overall results expressed as percentages for the Symmetry Task GSYM.

Score	Trial Schools (n=296)			Project schools (n=461)		
	GSYM1	GSYM2	GPSYM3	GSYM1	GSYM2	GSYM3
0	21.8	48.3	70	27.1	44.3	61.2
1	32.2	18.9	25.9	13.7	19.7	21.5
2	30.9	27.1	4.1	38.2	28.6	17.4
3	13.6	5.4		18.4	6.3	
4	1.6	0.3		2.6	1.1	

The data show a developmental progression in students' reasoning of symmetry (Table 2), with Year 5 performing slightly better than Year 4 in all three questions. Both Year 4 and 5 students were better able to identify reflectional (GSYM1) than rotational symmetry (GSYM3). There was also a reduction in the number of no response or irrelevant response for GSYM2 and GSYM3 between Year 4 and 5.

Table 2
Percentage breakdown for GSYM1, GSYM2, and GSYM3 according to year level.

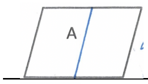
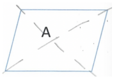
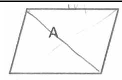
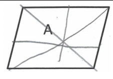
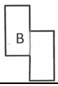
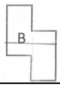
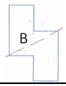
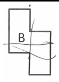
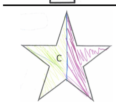
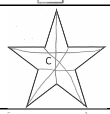
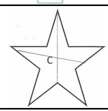
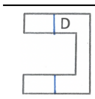
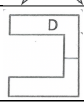
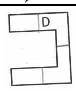
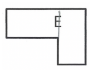
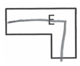
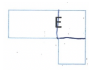
	Trial Schools						Project Schools			
Score	Yr4 *n*=31	Yr5 *n*= 59	Yr7 *n*= 55	Yr8 *n*=30	Yr9 *n*=81	Yr10 *n*=61	Yr7 *n*=210	Yr8 *n*=136	Yr9 *n*=47	Yr10 *n*=68
	GSYM1									
0	9.7	18.6	29.1	10	17.3	36.1	23.8	27.9	38.3	27.9
1	32.3	15.3	56.4	23.3	51.9	4.9	14.8	13.2	14.9	10.3
2	58.1	47.5	9.1	13.3	16.1	49.2	44.8	41.9	31.9	14.7
3	0	17	5.5	46.7	12.4	9.8	15.7	11	14.9	44.1
4	0	1.7	0	6.7	2.5	0	1	5.9	0	2.9
	GSYM2									
0	87.1	40.7	52.7	13.3	34.6	67.2	41.4	47.1	44.7	47.1
1	3.2	22	27.3	10	23.5	14.8	22.4	23.5	10.6	10.3
2	6.5	33.9	20	50	38.3	11.5	30.5	22.8	29.8	33.8
3	3.2	3.4	0	23.3	3.7	6.6	5.2	5.2	14.9	5.9
4	0	0	0	3.3	0	0	0.5	1.5	0	2.9
	GSYM3									
0	96.7	78	69.1	43.3	49.4	92	63.8	58.1	63.8	57.4
1	6.7	17	29.1	43.3	44.5	9.2	22.4	29.4	8.5	11.8
2	0	5.1	1.8	13.3	6.2	0	13.8	12.5	27.7	30.9

The results for Year 7 - 10 were mixed. In general, the project schools showed slight improvement when compared with their counterparts. For example, in GSYM1 Year 7, 15.7% project students correctly drew the lines of symmetry while only 5.5% Year 7 trial students did so. Year 8 trial students out performed all students on all tasks except for GSYM3. The small sample from one class may have influenced the result although data from other year levels were also collected from that same school. Differences in teacher knowledge and student engagement may have played a part in influencing student outcomes.

The use of artefacts and signs to reason about symmetry

In a semiotic process, both artefacts and signs interact to mediate a mental act. An artefact is related to a specific task and a specific mathematical knowledge. Signs are the product of an individual's internalisation of a mental function between the artefact and the task. A '/' sign drawn across angles may indicate diagonal or a line of symmetry. Their use is never neutral and must be interpreted within the context. Table 3 shows some of the most common signs found in the trial data, with item A having the most diverse range. Around 21% correct responses were made for item C and 70% for item D. Around 18% of the responses for item C were partially completed.

Table 3
Percentages of the most common signs used by trial schools for GSYM1.

	24		9.8		10.4		10.7		11
	26.8		6		7.3		5.4		
	43.8		5.4		17.7				
	1.9		70		4.7				
	22		3.8		3.2				

Bornstein et al. (1984) noted that the discrimination of symmetry progressed from vertical, to horizontal and lastly oblique. The trial students' experience with symmetry was rudimentary. The most common sign, found across all items except D, was a vertical line drawn in the middle. A smaller group drew one horizontal line across item A (3.8%), B (6%) and E (2.2%), and others drew an oblique line across item A (10.4%), B (7.3%), and E (2.2%). A number of students assumed that reflective symmetry means equal parts, others thought that it was synonymous to parallelism or equilateral (see Figure 3). A lack of understanding about the functions of mathematical notations in reasoning situations may have also led students to confuse lines of symmetry with diagonal lines. There are also incidences where students simply drew lines along the boundary of a shape.

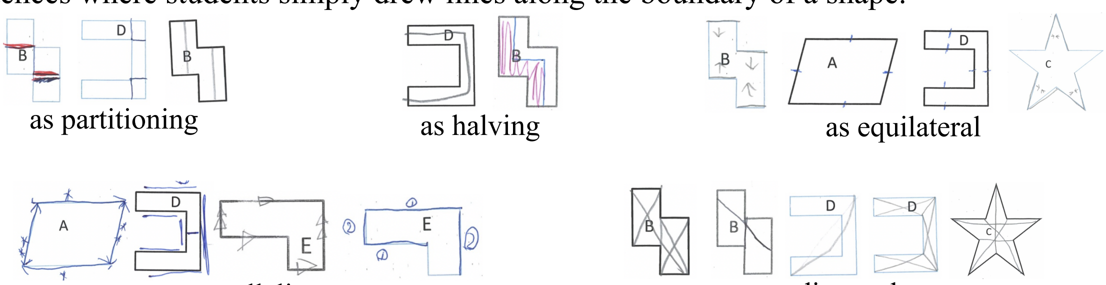


Figure 3. Signs made by trial students to indicate line symmetry.

GSYM2 was a difficult task for students to answer (Figure 4), as many were unable to comprehend what was required of them and did not respond. Others wrote their responses next to or under the criteria instead of in the specified space. We compared and analysed student's responses and awarded a score if there was no contradiction made. Although students often gave multiple entries for each category, only 3.5% were able to correctly identify item A and B as having rotational symmetry and not reflective symmetry.

	Has rotational symmetry	Does not have rotational symmetry
Has reflectional symmetry	C - 34.4%	D - 21.1%
Does not have reflectional symmetry	A - 9.8%, B 17.7%, AB 3.5%	E - 19.5%

Figure 4. Percentages of correct responses for GSYM2.

Of the 30% of trial students who responded in GSYM3, 23.7% have some idea that symmetry denotes proportionality. They also understood that some manipulation of an artefact is required, such as rotating (24.6%) flipping (1.9%), folding (3.2%) and turning (9.5%) to determine rotational symmetry. Around 4.7% alluded to the degrees of the angle but their description was not always clear.

Year 5: If you rotate the shape the lines of symmetry will be the same
Year 8: If you can rotate the shape and it looks the same all 360° around
Year 9: … because I imagine if I flip the shape or draw line the opposite way
Year 10: It could be folded from most or any angle.

There were 12 students who used diagrams to explain their reasoning. Their responses range from not understanding the difference between reflective and rotational symmetry (see Year 8 in Figure 5), partial understanding of rotation (Year 7 and Year 9A), to comprehending invariance during transformation (Year 9B). Although some responses alluded to the concept of partial turn as stated in the rubric, none were able to explain the concept based on the invariance of the centre of the turn. That is, understanding the centre of the turn in determining the angles of rotation.

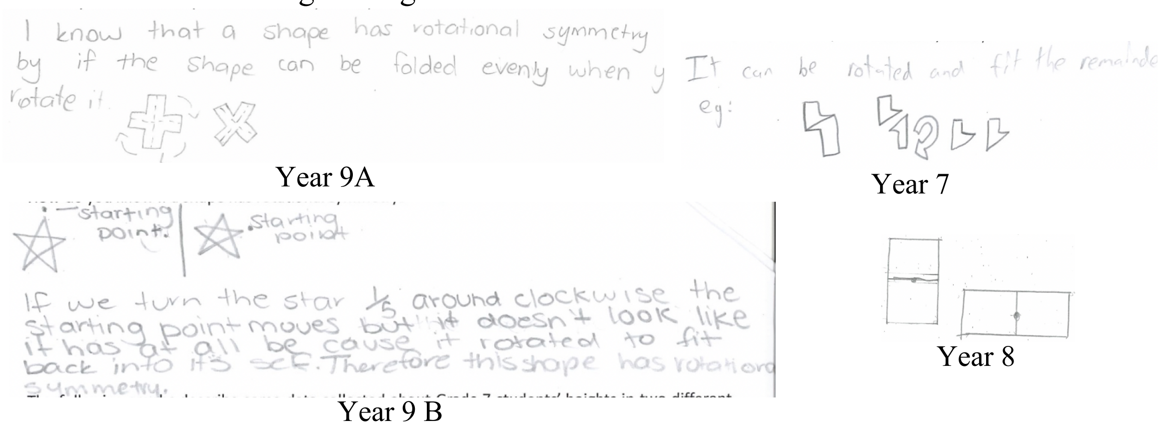


Figure 5. Examples of students' use of diagrams when reasoning about rotational symmetry.

Discussion

The results we obtained in this study showed a synergic relation between how students visualise the artefacts, and their understanding and use of signs (*keywords* and diagrams) in communicating their knowledge. Overall, students' knowledge of symmetry is fragmented, and their reasoning ability is poor. Although they can 'see' an element of symmetry in some of the artefacts, this knowledge is often confused with other mathematical ideas as shown in the signs they used. Thinking symmetry to mean 'equal half' may have caused some to partition the artefacts into equal parts. A lack of understanding of mathematical signs and convention resulted in students' invention of their own signs to show parallelism. Students also may not have understood the difference between diagonal lines and lines of symmetry. The poor results obtained on task GSYM2 reflected a lack of ability to systematically analyse all the information presented in the artefact in order to make deduction. Certainly, students need to be able to differentiate between reflective and rotational symmetry. This understanding needs to be built on knowing the concept of angle for without this, conceptual understanding of symmetry cannot be developed.

As stated earlier, this study is part of a larger study into developing a learning progression for mathematical reasoning. Future research can consider the development of a learning progression for rotational symmetry and factors that support its learning. Research should investigate how mediation of the semiotic processes can help students construct an understanding of symmetry, paying attention not only to the language and visualisation but also to the artefacts and signs that are used, making discussion of these much more explicit in the classroom. Specifically, investigation is needed about the type of tools (visual images as well as concrete objects) that support learning, the circumstances for mediation to make a difference, and the mediator's (the mathematics teacher) ability to make the concept accessible to students.

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